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Note

The Computation of Borel-Type Sums Arising in Scattering Theory

1. Introduction

It is often necessary, for instance in scattering theory [1], to calculate sums of the form

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n \tag{1}$$

for a wide range of values of the positive variable x, where $\{s_n\}$ is some fixed convergent sequence.

We use the notation

$$s_n \leftrightarrow f(x) \tag{2}$$

to indicate relationship (1) and we call f the *Borel transform* of the sequence $\{s_n\}$.

It is known that if

$$\lim_{n\to\infty} s_n = s \qquad \text{then} \qquad \lim_{x\to\infty} f(x) = s; \tag{3}$$

see Knopp, [2, p. 472]. From this point of view the relationship $s_n \leftrightarrow f$ is a summation process which can be used to compute the (generally unknown) value of the limit of the sequence s_n.

The problem presented by sums such as (1) when they occur in physics is usually the inverse of this: s_n is known (generally it is a correlation function) and the task is to compute the function f.

When x is small, the computational problems are not severe. When x is large, the computation of f from its defining series presents grave overflow–underflow problems, and the task is decidedly nontrivial. In many important cases, a technique for computing f may be obtained by asymptotic analysis.

In what follows we use the notation

$$s = \lim_{n\to\infty} s_n ,$$

$$s_n\hat{} = \sup_{k>n} | s_k | ,$$

$$r_n = s - s_n , \qquad \text{the remainder sequence,}$$

$$f_N(x) = s - e^{-x} \sum_{n=0}^{N} \frac{x^n r_n}{n!} , \tag{4}$$

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$$R_N(x) = -e^{-x} \sum_{n=N+1}^{\infty} \frac{x^n r_n}{n!}, \qquad \text{the remainder function,}$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \qquad n = 0, 1, 2,\ldots \text{ (Pochhammer's symbol).}$$

The notation for all special functions in this paper is that of [3].

By linearity of the "$\leftrightarrow$" relationship we have

$$\begin{aligned} f_N(x) + R_N(x) &= s - e^{-x} \sum_{n=0}^{\infty} \frac{x^n r_n}{n!} \\ &= s - e^{-x} \sum_{n=0}^{\infty} \frac{x^n (s - s_n)}{n!} \\ &= s - s + e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!} = f(x). \end{aligned} \tag{5}$$

2. Computation of f for x Small

If x is not too large, f_N is a good approximation to f for N suitably large. We have, in fact,

$$\begin{aligned} |f(x) - f_N(x)| = |R_N(x)| &\leqslant e^{-x} \sum_{n=N+1}^{\infty} \frac{x^n |r_n|}{n!} \\ &\leqslant r_N\hat{}\, e^{-x} \sum_{n=0}^{\infty} \frac{x^{N+n+1}}{(n+N+1)!}. \end{aligned} \tag{6}$$

Using the fact that

$$(u+v)! \geqslant u!\, v!, \tag{7}$$

we have

$$\begin{aligned} |f(x) - f_N(x)| &\leqslant \frac{r_N\hat{}\, e^{-x} x^{N+1}}{(N+1)!} \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= r_N\hat{}\, \delta_N(x), \\ \delta_N(x) &= \frac{x^{N+1}}{(N+1)!}. \end{aligned} \tag{8}$$

Thus for a given x we will have m decimal accuracy even for the most slowly convergent s_n if N is such that

$$x^{N+1}/(N+1)! < \tfrac{1}{2} \times 10^{-m-1}, \tag{9}$$

N suitably large. The use of Stirling's formula shows that we must have approximately

$$x < \frac{N+1}{e} \left\{ \left(\frac{\pi(N+1)}{2} \right)^{1/2} 10^{-m-1} \right\}^{1/(N+1)}. \tag{10}$$

Table I indicates how large x may be taken for a given accuracy and a given N.

TABLE I

Values of a for Given N and m[a]

	m							
N	3	4	5	6	7	8	9	10
10	2.0	1.6	1.3	1.0	0.8	0.7		
15	3.6	3.1	2.7	2.3	2.0	1.7	1.5	
20	5.4	4.8	4.3	3.8	3.4	3.1	2.8	2.5
30	9.0	8.3	7.7	7.2	6.7	6.2	5.7	5.3
50	16.3	15.6	14.9	14.3	13.6	13.0	12.5	11.9
70	23.7	23.0	22.2	21.5	20.8	20.2	19.5	18.9
100	34.8	34.0	33.2	32.5	31.7	31.0	30.3	29.6

[a] To compute $f(x)$ to m-digit accuracy using $f_N(x)$ take $x < a$.

3. Large x

The sequence s_n often has an asymptotic or convergent representation of the form

$$s_n \sim s + \lambda^n \left[\frac{C_1}{n} + \frac{C_2}{n^2} + \cdots \right], \qquad \lambda \neq 0, |\lambda| \leqslant 1, n \to \infty. \tag{11}$$

In such cases, an asymptotic representation may be obtained for $f(x)$ as $x \to \infty$.

To start, we seek to determine the Borel transform of a simple sequence, $\lambda^n/(n+a)_k$, $a > 0$, $k = 1, 2, \ldots$.

$$\frac{\lambda^n}{(n+a)_k} \leftrightarrow \frac{e^{-x}}{(a)_k} \sum_{n=0}^{\infty} \frac{(x\lambda)^n (a)_n}{n!(a+k)_n} = \frac{e^{-x}}{(a)_k} \Phi(a, a+k, x\lambda) = f^{(k)}(x). \tag{12}$$

For large x, $f^{(k)}(x)$ has the asymptotic behavior

$$f^{(k)}(x) \sim \frac{e^{x(\lambda-1)}}{(x\lambda)^k} \sum_{r=0}^{\infty} \frac{(k)_r (1-a)_r (x\lambda)^{-r}}{r!} + \frac{\Gamma(a)\, e^{-x} \cos(\pi a)}{\Gamma(k)(x\lambda)^a} \sum_{r=0}^{k-1} \frac{(a)_r (1-k)_r}{r!} (-1)^r (x\lambda)^{-r}, \quad x \to \infty, \tag{13}$$

see [3, Vol. I, p. 278]. Note that the second term above is finite (convergent).

The most important case is the case when $a = 1$. Then all terms but the first of the first sum vanish and we have the exact representation

$$\frac{\lambda^n}{(n+1)_k} \leftrightarrow \frac{e^{x(\lambda-1)}}{(x\lambda)^k} + V_k, \qquad V_k = \frac{-e^{-x}}{\Gamma(k)} \sum_{r=0}^{k-1} (1-k)_r (-1)^r (x\lambda)^{-r-1} = O\left(\frac{e^{-x}}{x}\right), \qquad x \to \infty. \tag{14}$$

We now use the fact that

$$\sum_{s=k}^{\infty} \frac{A_{k,s}}{(n+1)_s} = \frac{1}{n^k}, \qquad n > 0, \tag{15}$$

where the $A_{k,s}$ may be written in terms of the generalized Bernoulli polynomials as follows (see Table II)

$$A_{k,s} = \binom{s-1}{k-1} B_{s-k}^{(s)}(s), \qquad k \leqslant s,\ s = 1, 2, 3, \ldots. \tag{16}$$

See Nörlund [4, p. 261]. $A_{k,s}$ can be conveniently calculated from

$$A_{k,s} = \text{coefficient of } x^{k-1} \text{ in } (x+1)(x+2)\cdots(x+s-1), \qquad s = 1, 2, 3, \ldots. \tag{17}$$

See [4, p. 147]. Thus

$$\lambda^n/n^k \leftrightarrow e^{x(\lambda-1)} \sum_{s=k}^{\infty} A_{k,s}(x\lambda)^{-k} + U_k, \qquad U_k = O(e^{-x}/x), \tag{18}$$

TABLE II

$A_{k,s}$

	k						
s	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
6	120	274	225	85	15	1	
7	720	1764	1624	735	175	21	1

the series being an asymptotic series. (The above estimates can easily be justified in our result (19)–(20) below by first assuming that s_n is a series of terms $\lambda^n/(n+1)_k$ and then rearranging in terms λ^n/n^k. The present computations seem to be more straightforward.)

Now using representation (11) and invoking the linearity of the Borel transform we find that if

$$s_n \sim s + \lambda^n \left[\frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\right], \qquad n \to \infty, \tag{19}$$

then

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!} \sim s + e^{x(\lambda-1)} \left[\frac{\bar{C}_1}{x} + \frac{\bar{C}_2}{x^2} + \frac{\bar{C}_3}{x^3} + \cdots\right], \qquad x \to \infty, \tag{20}$$

where

$$\bar{C}_s = \lambda^{-s} \sum_{k=1}^{s} A_{k,s} C_k ,$$

$$\bar{C}_1 = \frac{C_1}{\lambda}, \qquad \bar{C}_2 = \frac{C_1 + C_2}{\lambda^2}, \qquad \bar{C}_3 = \frac{2C_1 + 3C_2 + C_3}{\lambda^3}, \tag{21}$$

$$\bar{C}_6 = \frac{6C_1 + 11C_2 + 6C_3 + C_6}{\lambda^4}, \ldots .$$

When s_n has a known factorial series development

$$s_n = s + \lambda^n \left[\frac{D_1}{(n+1)} + \frac{D_2}{(n+1)(n+2)} + \cdots\right] \tag{22}$$

with, say,

$$\overline{\lim_{k\to\infty}} \mid D_k \mid^{1/k} < \sigma \tag{23}$$

then all series are convergent and we have

$$s_n \leftrightarrow s + e^{x(\lambda-1)} \sum_{k=1}^{\infty} D_k (x\lambda)^{-k} - e^{-x} \sum_{k=1}^{\infty} D_k^* (x\lambda)^{-k},$$

$$D_k^* = \sum_{r=0}^{\infty} \frac{D_{k+r}}{r!}, \qquad \mid x\lambda \mid > \sigma. \tag{24}$$

An interesting case is the case when $D_k = (-\beta)^k$, $\beta > 0$. Then

$$s_n \leftrightarrow s - \frac{\beta e^{-x}(e^{\lambda x} - e^{-\beta})}{(\beta + x\lambda)}, \tag{25}$$

and setting $\lambda = 1$ gives

$$\Phi(1, n+1, -\beta) \leftrightarrow (x + \beta e^{-x-\beta})/(x + \beta). \tag{26}$$

This Borel transform has a close relationship to some transforms occurring in turbulent scattering theory.

4. Examples

Consider the following incoherent scattering function for a surface with an exponential correlation function:

$$\phi(\alpha, \beta) = e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!\, n^2} \left[1 + \frac{\beta}{n^2}\right]^{-3/2},$$

$$\alpha = [k\sigma(\cos\theta_0 + \cos\theta)]^2, \qquad (27)$$

$$\beta = [ka(\sin\theta_0 + \sin\theta)]^2$$

where θ_0 is the angle the position vector of the transmitter makes with the vertical, θ the angle the position vector of the receiver makes with the vertical, σ the radar cross section, k the radar wavelength, and a the correlation length of wave.

$$s_n = \frac{[1 + (\beta/n^2)]^{-3/2}}{n^2} = \frac{1}{n^2} - \frac{\frac{3}{2}\beta}{n^4} + \cdots. \qquad (28)$$

We have $\lambda = 1$, $s = 0$, and

$$C_1 = C_3 = C_5 = \cdots = 0,$$

$$C_2 = 1, \qquad C_4 = \frac{-3}{2}\beta, \qquad C_6 = \frac{15}{8}\beta^2, \qquad C_8 = \frac{-35}{16}\beta^3, \ldots \qquad (29)$$

Thus

$$\phi(\alpha, \beta) \sim \frac{1}{\alpha^2} + \frac{3}{\alpha^3} + \frac{11 - \frac{3}{2}\beta}{\alpha^4} + \frac{50 - 15\beta}{\alpha^5} + \cdots, \qquad \alpha \to \infty. \qquad (30)$$

For $\alpha = 10$, $\beta = 1$ the terms above give 0.01430 with an error 2×10^{-5}. Notice the expansion is not uniform in β and the accuracy deteriorates with increasing β. In any case, a good policy for computing from asymptotic expansions is to stop before the smallest term; see Knopp [2].

Next consider the incoherent scattering function for a surface with a Gaussian correlation function

$$\Psi(\alpha, \beta) = e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left[\frac{e^{-\beta/n}}{n}\right], \qquad (31)$$

where α, β are as before except a is to be replaced by $a/2$ in (27). Then

$$s_n = \frac{e^{-\beta/n}}{n} = \frac{1}{n} - \frac{\beta}{n^2} + \frac{\beta^2/2}{n^3} - \frac{\beta^3/6}{n^4} + \cdots, \qquad (32)$$

so again $\lambda = 1$ and

$$\Psi(\alpha, \beta) \sim \frac{1}{\alpha} + \frac{1 - \beta}{\alpha^2} + \frac{(\beta^2 - 6\beta + 4)}{2\alpha^3} - \frac{(\beta^3 - 18\beta^2 + 66\beta - 36)}{6\alpha^4} + \cdots, \qquad \alpha \to \infty. \qquad (33)$$

With $\beta = \frac{1}{2}, \alpha = 10$ the four terms above give $\Psi = 0.1057$ with an error of less than one-half unit in the last decimal place.

5. Comments

The transform pair given in Section 2,

$$s_n = \Phi(1, n+1; -\beta) \leftrightarrow (x + \beta e^{-x-\beta})/(x+\beta) = f(x), \tag{34}$$

has some of the characteristics of the Gaussian correlation transform pair

$$t_n = e^{-\beta/n}/n \leftrightarrow g(x). \tag{35}$$

For β large and $x \ll \beta$, f and g are exponentially small in x. Nevertheless, f and g ultimately behave algebraically in x, $f = 1 + o(1)$, $g = (1/x)[1 + o(1)]$ as $x \to \infty$. Thus there is a transitional x-region in which f and g move from exponential behavior to algebraic behavior.

The graph of g given in [1] reflects this, the graph becoming increasingly steep as β increases in the neighborhood of $x = 10$.

References

1. J. Jarem, "Scattering from a Turbulent Overdense Surface," Technical Monograph No. 64-08, RCA Missile and Surface Radar Division, Moorestown, N.J., December 1964.
2. K. Knopp, "Theory and Application of Infinite Series," Hafner, New York, 1951.
3. Erdélyi (Ed.) "Higher Transcendental Functions," 3 vol., McGraw-Hill, New York, 1953.
4. N. E. Nörlund, "Vorlesungen über Differenzenrechnung," Chelsea, New York, 1954.

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